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The Discrete Coagulation Equations with Collisional Breakage

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The discrete coagulation equations with collisional breakage describe the dynamics of cluster growth when clusters undergo binary collisions resulting either in coalescence or breakup with possible transfer of matter. Each of these two events may happen with an *a priori* prescribed probability depending for instance on the sizes of the colliding clusters. We study the existence, density conservation and uniqueness of solutions. We also consider the large time behaviour and discuss the possibility of the occurrence of gelation in some particular cases.

KEY WORDS: Cluster growth; coalescence; collisional breakage; existence of solutions; propagation of moments.

1. INTRODUCTION

Coagulation-fragmentation processes naturally occur in the dynamics of cluster growth and describe the way a system of clusters can merge to form larger ones or fragment to form smaller ones. Models of cluster growth arise in a wide variety of situations, including aerosol science, astrophysics, colloidal chemistry, polymer science, and biology. In the model considered in this paper the clusters are assumed to be discrete, that is, they consist of a finite number of identical elementary particles. The basic reactions between clusters taken into account are the coalescence of two clusters to form a larger one and the breakage of clusters into smaller pieces. At least two physical mechanisms have been considered to describe the latter

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process. The most commonly used, known as *spontaneous* or *linear* fragmentation, assumes that the breakup process is only ruled by the properties of the particles (and also by external forces, if any). The rate of fragmentation of clusters made of i particles, or i-clusters, is then taken to be proportional to the number of i-clusters per unit volume (hence the term linear used for this kind of fragmentation). The *collisional* or *nonlinear* fragmentation process is based on a different assumption, namely that the breakage of a cluster only occurs after collision with another cluster, the rate of this reaction being taken to be proportional to the numbers per unit volume of the two colliding clusters. Let us point out here one main difference between these two fragmentation processes. The spontaneous breakage of a cluster only produces smaller clusters while collisional breakage allows for some transfer of matter between the two colliding clusters and might thus produce clusters which are larger than the two colliding ones. For example, the collisional breakage of an i-cluster and a j-cluster might result in a 1-cluster and an $(i+j-1)$-cluster.

Denoting by $c_i(t)$, $i \geqslant 1$, the number of i-clusters per unit volume at time $t \geqslant 0$, the discrete coagulation equations with spontaneous fragmentation read

$$\frac{dc_i}{dt} = \frac{1}{2} \sum_{j=1}^{i-1} (a_{j,i-j} c_j c_{i-j} - b_{j,i-j} c_i) - \sum_{j=1}^{\infty} (a_{i,j} c_i c_j - b_{i,j} c_{i+j}), \qquad (1.1)$$

for $i \geqslant 1$, under the additional assumption that only binary fragmentation is allowed for (for $i=1$ the first sum of the right-hand side of (1.1) is obviously taken to be zero). Here $(a_{i,j})$ and $(b_{i,j})$ denote the coagulation and fragmentation coefficients, respectively, and satisfy

$$a_{i,j} = a_{j,i} \geqslant 0 \qquad \text{and} \qquad b_{i,j} = b_{j,i} \geqslant 0, \qquad i, j \geqslant 1.$$

In the right-hand side of (1.1) the first term accounts for the formation of i-clusters by binary coalescence of smaller ones and the second one for the fragmentation of i-clusters into two smaller ones. The third term describes the depletion of i-clusters by coagulation with other clusters while the fourth term represents the creation of i-clusters resulting from the breakage of larger ones. The system (1.1) without fragmentation ($b_{i,j} \equiv 0$) was originally introduced by Smoluchowski[21, 22] and we refer to Drake[11] for a derivation of (1.1) and some physical background. In the past years several mathematical studies have been devoted to (1.1) and we refer among others to refs. 23, 3, 2, 6, and 15 for existence and uniqueness results. The large time behaviour of solutions to (1.1) has also been investigated for some particular choices of the coefficients in, e.g., refs. 3–5 (see also the survey paper by da Costa[7]).

The modeling of collisional breakage requires a different formulation. More precisely, the discrete coagulation equations with collisional fragmentation read[20, 26]

$$\frac{dc_i}{dt} = \frac{1}{2} \sum_{j=1}^{i-1} w_{j,i-j} a_{j,i-j} c_j c_{i-j} - \sum_{j=1}^{\infty} a_{i,j} c_i c_j \qquad (1.2)$$

$$+ \frac{1}{2} \sum_{j=i+1}^{\infty} \sum_{k=1}^{j-1} N^i_{j-k,k} (1 - w_{j-k,k})\, a_{j-k,k} c_{j-k} c_k,$$

$$c_i(0) = c_i^0, \qquad (1.3)$$

for $i \geqslant 1$. Here $a_{i,j}$ denotes the rate of collisions of i-clusters with j-clusters and $w_{i,j}$ is the probability that the two colliding clusters merge into a single one. If they do not (an event which occurs with probability $1 - w_{i,j}$) they undergo fragmentation with possible transfer of matter. Then $\{N^s_{i,j}, s = 1,..., i+j-1\}$ is the distribution function of the resulting fragments. The coefficients $(a_{i,j})$, $(w_{i,j})$ and $(N^s_{i,j})$ enjoy the following properties:

$$a_{i,j} = a_{j,i} \geqslant 0 \qquad \text{and} \qquad 0 \leqslant w_{i,j} = w_{j,i} \leqslant 1, \qquad i, j \geqslant 1, \qquad (1.4)$$

and as mass is required to be conserved during each collision,

$$N^s_{i,j} = N^s_{j,i} \geqslant 0 \qquad \text{and} \qquad \sum_{s=1}^{i+j-1} s N^s_{i,j} = i + j, \qquad i, j \geqslant 1. \qquad (1.5)$$

In the right-hand side of (1.2) the first term accounts for the formation of i-clusters by collision and coagulation of smaller ones (with effective rate $w_{i,j} a_{i,j}$) and the second one for the loss of i-clusters due to collisions with other clusters. The third term describes the creation of i-clusters after the collision and breakup of larger clusters.

In contrast to (1.1) the system (1.2) does not seem to have been investigated mathematically and the purpose of this paper is to discuss some mathematical issues for (1.2). Let us first point out that, in the absence of fragmentation ($w_{i,j} \equiv 1$), the system (1.2) is nothing but the classical coagulation equation[21] which has been extensively studied by physicists and mathematicians (see, e.g., the survey paper by Aldous[1]). Observe next that, since particles are neither created nor destroyed in the reactions described by (1.1) or (1.2), the density

$$\varrho(t) = \sum_{i=1}^{\infty} i c_i(t) \qquad (1.6)$$

is expected to be conserved throughout time evolution. It is however well-known by now that, in the absence of fragmentation (i.e., $b_{i,j} \equiv 0$ for (1.1) or $w_{i,j} \equiv 1$ for (1.2)), there are physically relevant coefficients $(a_{i,j})$ for which density conservation breaks down in finite time, a phenomenon known as gelation (see, e.g., refs. 25, 12 and the references therein). It is also known that for (1.1) strong fragmentation prevents the gelation phenomenon to occur.[6] The gelation phenomenon might also take place for (1.2) as we shall see in Section 4.

Some particular cases of (1.2) have been considered by physicists and we mention some of them now. Besides the classical coagulation equation which is obtained from (1.2) by setting $w_{i,j} \equiv 1$, we can also consider the case where the collision of an i-cluster and a j-cluster results in either the coalescence of both in an $(i+j)$-cluster or in an elastic collision leaving the incoming clusters unchanged. In that case we have $N_{i,j}^i = N_{i,j}^j = 1$ and $N_{i,j}^s = 0$ if $s \notin \{i, j\}$. The system (1.2) then reduces to the classical coagulation equation with coagulation coefficients $(w_{i,j} a_{i,j})$, i.e.,

$$\frac{dc_i}{dt} = \frac{1}{2} \sum_{j=1}^{i-1} w_{j,i-j} a_{j,i-j} c_j c_{i-j} - \sum_{j=1}^{\infty} w_{i,j} a_{i,j} c_i c_j, \qquad i \geqslant 1.$$

As for models involving only collisional fragmentation $(w_{i,j} \equiv 0)$ we mention the nonlinear breakage model studied by Cheng and Redner.[8] In this model, when two clusters collide, they both fragment into smaller pieces and there is thus no transfer of matter between the colliding clusters. Actually the model studied in ref. 8 belongs to the class of continuous models, in which clusters are described by means of a continuous variable (volume or size). For clusters described by a discrete variable it reads

$$\frac{dc_i}{dt} = \sum_{j=1}^{\infty} \sum_{k=i+1}^{\infty} K_{j,k} b_{i,j;k} c_j c_k - \sum_{j=1}^{\infty} K_{i,j} c_i c_j, \qquad i \geqslant 1, \tag{1.7}$$

with $K_{i,j} = K_{j,k} \geqslant 0$, and $\{b_{i,j;k},\ 1 \leqslant i \leqslant j-1\}$ denotes the distribution function of the fragments of a j-cluster after a collision with a k-cluster, and satisfies

$$\sum_{i=1}^{j-1} i b_{i,j;k} = j, \qquad j \geqslant 2,\ k \geqslant 1.$$

To obtain (1.7) from (1.2) we put $a_{i,j} = K_{i,j}$ and

$$N_{i,j}^s = \mathbf{1}_{[s,+\infty)}(i)\, b_{s,i;j} + \mathbf{1}_{[s,+\infty)}(j)\, b_{s,j;i}$$

for $i, j \geqslant 1$ and $s \in \{1,..., i+j-1\}$, where $\mathbf{1}_{[s,+\infty)}$ denotes the characteristic function of the interval $[s, +\infty)$. As each cluster splits into smaller pieces after collision it is expected that, in the long time, only 1-clusters remain and this is shown in Section 4. Large time asymptotics for the continuous analogue of (1.7) may be found in refs. 8, 9, and 14. Finally a particular case of (1.2) was used by Srivastava[24] to analyse the evolution of raindrops size spectra and reads

$$\frac{dc_i}{dt} = \frac{1}{2} \sum_{j=1}^{i-1} K_{j,i-j} c_j c_{i-j} - \sum_{j=1}^{\infty} (K_{i,j} + \beta_{i,j})\, c_i c_j, \qquad i \geqslant 2,$$

$$\frac{dc_1}{dt} = -\sum_{j=1}^{\infty} (K_{1,j} + \beta_{1,j})\, c_1 c_j + \frac{1}{2} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} (j+k)\, \beta_{j,k} c_j c_k,$$

where $K_{i,j} = K_{j,i} \geqslant 0$ and $\beta_{i,j} = \beta_{j,i} \geqslant 0$. Introducing

$$a_{i,j} = K_{i,j} + \beta_{i,j}, \qquad w_{i,j} = \frac{K_{i,j}}{a_{i,j}}, \qquad N^s_{i,j} = (i+j)\, \delta_{s,1}$$

allows to check that the previous system is a particular case of (1.2). Assuming that both $K_{i,j}$ and $\beta_{i,j}$ are constants an explicit solution is obtained which converges to a steady state as time goes to infinity.[24]

We now describe the results we obtain in this paper: we mainly discuss the existence and uniqueness of solutions to (1.2), though the final section is devoted to the study of the large time behaviour for some particular cases. In the next section we prove the existence of solutions to (1.2) under rather general assumptions on the coefficients $(a_{i,j})$, $(w_{i,j})$ and $(N^s_{i,j})$. As our assumptions include the classical coagulation equations with coagulation coefficients $(a_{i,j})$ for which gelation is known to occur, we only prove that the density of the solutions is non-increasing with respect to time. In Section 3 we consider the existence of density-conserving solutions to (1.2) and prove the existence of such a solution when $a_{i,j} \leqslant A(i+j)$. A similar result has been proved for (1.1) in ref. 2 but our proof relies on a completely different argument which is adapted from ref. 16. It involves the study of the propagation of generalised moments of approximating solutions to (1.2) without additional assumptions on the initial data. Still assuming the collision coefficients $(a_{i,j})$ to be at most linear we investigate the propagation of moments for the density-conserving solutions we construct, and their uniqueness as well. Finally the large time behaviour of solutions to (1.2) seems to be a challenging question. Still, in a few particular cases, we are able to prove the stabilization to steady states and these results are described in the last section. We also point out the possible occurrence of gelation in the model (1.2).

From now on we assume that the coefficients $(a_{i,j})$, $(w_{i,j})$ and $(N^s_{i,j})$ are given and satisfy (1.4)–(1.5).

2. EXISTENCE OF SOLUTIONS

We first introduce some notations and specify what we mean by a solution to (1.2)–(1.3). Obviously the density defined by (1.6) is a relevant quantity for the analysis of (1.2)–(1.3) and a natural functional setting is given by the Banach space X defined by

$$X = \left\{ x = (x_i)_{i \geqslant 1} \in \mathbb{R}^{\mathbb{N}\setminus\{0\}},\ \sum_{i=1}^{\infty} i\,|x_i| < \infty \right\},$$

with the norm

$$\|x\|_X = \sum_{i=1}^{\infty} i\,|x_i|.$$

We shall actually use the positive cone X^+ of X, that is,

$$X^+ = \{x \in X,\ x_i \geqslant 0 \text{ for each } i \geqslant 1\}.$$

Next, if $x = (x_i)_{i \geqslant 1}$ is a sequence of real numbers and i, j are positive integers we put

$$D^s_{i,j}(x) = N^s_{i,j}(1 - w_{i,j})\, a_{i,j} x_i x_j, \qquad 1 \leqslant s \leqslant i+j-1. \tag{2.1}$$

Definition 1. Let $T \in (0, +\infty]$ and $c^0 = (c^0_i)_{i \geqslant 1}$ be a sequence of non-negative real numbers. A solution $c = (c_i)_{i \geqslant 1}$ to (1.2)–(1.3) on $[0, T)$ is a sequence of non-negative continuous functions satisfying for each $i \geqslant 1$ and $t \in (0, T)$

(i) $c_i \in \mathscr{C}([0, T))$, $\sum_{j=1}^{\infty} a_{i,j} c_j \in L^1(0, t)$, $\sum_{j=i+1}^{\infty} \sum_{k=1}^{j-1} D^i_{j-k,k}(c) \in L^1(0, t)$,

(ii) and there holds

$$c_i(t) = c^0_i + \int_0^t \left(\frac{1}{2} \sum_{j=1}^{i-1} w_{j,i-j} a_{j,i-j} c_j(\tau)\, c_{i-j}(\tau) - \sum_{j=1}^{\infty} a_{i,j} c_i(\tau)\, c_j(\tau) \right) d\tau$$
$$+ \frac{1}{2} \int_0^t \sum_{j=i+1}^{\infty} \sum_{k=1}^{j-1} D^i_{j-k,k}(c(\tau))\, d\tau.$$

We now fix a sequence $c^0 = (c_i^0)_{i \geqslant 1}$ of non-negative real numbers as the initial condition. As in previous works on similar equations existence of solutions to (1.2)–(1.3) follows by taking a limit of solutions to finite-dimensional systems of ordinary differential equations obtained by truncation of (1.2). More precisely, given $N \geqslant 3$, we consider the following system of N ordinary differential equations

$$\frac{dc_i^N}{dt} = \frac{1}{2} \sum_{j=1}^{i-1} w_{j,i-j} a_{j,i-j} c_j^N c_{i-j}^N - \sum_{j=1}^{N-i} a_{i,j} c_i^N c_j^N + \frac{1}{2} \sum_{j=i+1}^{N} \sum_{k=1}^{j-1} N_{j-k,k}^i (1 - w_{j-k,k})\, a_{j-k,k} c_{j-k}^N c_k^N, \qquad (2.2)$$

$$c_i^N(0) = c_i^0, \qquad (2.3)$$

for $i \in \{1,..., N\}$. Proceeding as in ref. 2, Lemmas 2.1 and 2.2 we obtain the following result.

Lemma 2.2. For each $N \geqslant 3$ the system (2.2)–(2.3) has a unique solution

$$c^N = (c_i^N)_{1 \leqslant i \leqslant N} \in \mathscr{C}^1([0, +\infty); \mathbb{R}^N)$$

with $c_i^N(t) \geqslant 0$ for $1 \leqslant i \leqslant N$ and $t \geqslant 0$. Furthermore, if $(g_i) \in \mathbb{R}^N$, there holds

$$\sum_{i=1}^{N} g_i \frac{dc_i^N}{dt} = \frac{1}{2} \sum_{i=1}^{N-1} \sum_{j=1}^{N-i} (g_{i+j} - g_i - g_j)\, a_{i,j} c_i^N c_j^N - \frac{1}{2} \sum_{i=1}^{N-1} \sum_{j=1}^{N-i} (1 - w_{i,j}) \left(g_{i+j} - \sum_{s=1}^{i+j-1} N_{i,j}^s g_s \right) a_{i,j} c_i^N c_j^N. \qquad (2.4)$$

It easily follows from (1.5) and (2.4) with $g_i = i$, $1 \leqslant i \leqslant N$, that

$$\sum_{i=1}^{N} i c_i^N(t) = \sum_{i=1}^{N} i c_i^0, \qquad t \in [0, +\infty). \qquad (2.5)$$

We now state the main result of this section.

Theorem 2.3. Assume that $c^0 \in X^+$ and the assumptions (1.4)–(1.5) are fulfilled. Assume further that

$$\lim_{i \to +\infty} \max_{1 \leqslant j \leqslant i-1} \left(\frac{a_{i-j,j}}{j(i-j)} \right) = 0, \qquad (2.6)$$

and there is a constant C_1 such that

$$N^s_{i,j} \leqslant C_1, \qquad 1 \leqslant s \leqslant i+j-1, \qquad i, j \geqslant 1. \tag{2.7}$$

Then there is at least one solution c to (1.2)–(1.3) on $[0, +\infty)$ which satisfies $c(t) \in X^+$ for each $t \in [0, +\infty)$ and

$$\sum_{i=1}^{\infty} ic_i(t) \leqslant \sum_{i=1}^{\infty} ic_i^0. \tag{2.8}$$

Remark 2.4. It is easily seen that the collision coefficients $a_{i,j} = i^\alpha j^\alpha$, $i, j \geqslant 1$, fulfil (2.6) when $\alpha \in [0, 1)$.

Proof. We first notice that (2.6) ensures that there is a positive constant C_0 such that

$$a_{i,j} \leqslant C_0 ij, \qquad i, j \geqslant 1. \tag{2.9}$$

It also follows from (2.6) that, for each $i \geqslant 1$,

$$\lim_{j \to +\infty} \frac{a_{i,j}}{j} = 0. \tag{2.10}$$

Fix $T \in (0, +\infty)$. In the following we denote by C any positive constant depending only on C_1, C_0, $\|c^0\|_X$ and T. Consider now $i \geqslant 1$ and $N \geqslant i$. It follows from Lemma 2.2, (1.4), (2.5), (2.7) and (2.9) that the ith component c_i^N of the solution c^N to (2.2)–(2.3) satisfies

$$\begin{aligned}\left|\frac{dc_i^N}{dt}\right| &\leqslant \frac{C_0}{2} \sum_{j=1}^{i-1} j(i-j)\, c_{i-j}^N c_j^N + C_0 \sum_{j=1}^{N-i} ijc_i^N c_j^N \\ &\quad + \frac{C_0 C_1}{2} \sum_{j=i+1}^{N} \sum_{k=1}^{j-1} (j-k)\, kc_{j-k}^N c_k^N \\ &\leqslant C\, \|c^0\|_X^2 \\ \left|\frac{dc_i^N}{dt}\right| &\leqslant C. \end{aligned} \tag{2.11}$$

By (2.5) and (2.11) the sequence $(c_i^N)_{N \geqslant i}$ is bounded in $\mathscr{C}^1([0, T])$ and is thus relatively compact in $\mathscr{C}([0, T])$. As this holds true for each $i \geqslant 1$ we use a diagonal process to conclude that there is a subsequence of (c^N) (not relabeled) and a sequence $c=(c_i)_{i\geqslant 1}$ of non-negative continuous functions such that, for each $i\geqslant 1$,

$$\lim_{N\to+\infty} |c_i^N - c_i|_{\mathscr{C}([0,T])} = 0. \tag{2.12}$$

In addition we infer from (2.5) and (2.12) that, for each $M\geqslant 1$ and $t\in[0,T]$, we have

$$\sum_{i=1}^{M} i c_i(t) \leqslant \sum_{i=1}^{\infty} i c_i^0,$$

hence

$$\sum_{i=1}^{\infty} i c_i(t) \leqslant \sum_{i=1}^{\infty} i c_i^0, \qquad t\in[0,T]. \tag{2.13}$$

We next fix $i\geqslant 1$ and consider $\varepsilon\in(0,1)$. By (2.10) there is $M\geqslant 1$ such that $a_{i,j}\leqslant \varepsilon j$ for $j\geqslant M$. For $t\in[0,T]$ and N large enough it follows from (2.5) and (2.13) that

$$\begin{aligned}
&\left|\sum_{j=1}^{N-i} a_{i,j} c_j^N(t) - \sum_{j=1}^{\infty} a_{i,j} c_j(t)\right| \\
&\quad \leqslant C_0 i \sum_{j=1}^{M} j\,|c_j^N(t)-c_j(t)| + \varepsilon \sum_{j=M+1}^{N-i} j c_j^N(t) + \varepsilon \sum_{j=M+1}^{\infty} j c_j(t) \\
&\quad \leqslant C_0 i \sum_{j=1}^{M} j\,|c_j^N - c_j|_{\mathscr{C}([0,T])} + 2\varepsilon\,\|c^0\|_X.
\end{aligned}$$

We then infer from (2.12) that

$$\limsup_{N\to+\infty} \left|\sum_{j=1}^{N-i} a_{i,j} c_j^N - \sum_{j=1}^{\infty} a_{i,j} c_j\right|_{\mathscr{C}([0,T])} \leqslant 2\varepsilon\,\|c^0\|_X,$$

from which we conclude that

$$\lim_{N\to+\infty} \left|\sum_{j=1}^{N-i} a_{i,j} c_j^N - \sum_{j=1}^{\infty} a_{i,j} c_j\right|_{\mathscr{C}([0,T])} = 0.$$

Using again (2.12) we end up with

$$\lim_{N\to+\infty} \left|\sum_{j=1}^{N-i} a_{i,j} c_i^N c_j^N - \sum_{j=1}^{\infty} a_{i,j} c_i c_j\right|_{\mathscr{C}([0,T])} = 0. \tag{2.14}$$

We next proceed in a similar way to show that (2.6) entails

$$\lim_{N\to+\infty} \left\| \sum_{j=i+1}^{N} \sum_{k=1}^{j-1} D^i_{j-k,k}(c^N) - \sum_{j=i+1}^{\infty} \sum_{k=1}^{j-1} D^i_{j-k,k}(c) \right\|_{\mathscr{C}([0,T])} = 0, \tag{2.15}$$

where $D^i_{j-k,k}(.)$ is defined by (2.1). Indeed let $\varepsilon \in (0,1)$. By (2.6) there is $M \geqslant i+1$ such that

$$\max_{1\leqslant k\leqslant j-1} \left(\frac{a_{j-k,k}}{k(j-k)} \right) \leqslant \varepsilon, \qquad j \geqslant M. \tag{2.16}$$

It follows from (1.4), (2.5), (2.7) and (2.16) that, for $N > M$,

$$\sup_{t\in[0,T]} \sum_{j=M}^{N} \sum_{k=1}^{j-1} D^i_{j-k,k}(c^N(t)) \leqslant C_1 \varepsilon \sup_{t\in[0,T]} \sum_{j=M}^{N} \sum_{k=1}^{j-1} (j-k)\, k c^N_{j-k}(t)\, c^N_k(t)$$
$$\leqslant C\varepsilon.$$

Similarly we infer from (1.4), (2.7), (2.13) and (2.16) that

$$\sup_{t\in[0,T]} \sum_{j=M}^{\infty} \sum_{k=1}^{j-1} D^i_{j-k,k}((c(t)) \leqslant C\varepsilon.$$

Combining the above two estimates and (2.12) yield the claim (2.15).

Thanks to (2.12), (2.14) and (2.15) it is now straightforward to pass to the limit as $N \to +\infty$ in the integral version of (2.2) and check that c is indeed a solution to (1.2)–(1.3) on $[0,T)$. Recalling (2.13) we see that c satisfies (2.8). As T was arbitrary, the proof of Theorem 2.3 is complete. ∎

Remark 2.5. As already mentioned it is in general not possible to improve (2.8) to an equality without additional assumptions on the data. A sufficient condition on $(a_{i,j})$ which guarantees the existence of a density-conserving solution is given in the next section. We will however return shortly to the gelation phenomenon in the final section.

3. DENSITY-CONSERVING SOLUTIONS

In this section we assume that the collision coefficients $(a_{i,j})$ satisfy

$$a_{i,j} \leqslant A(i+j), \qquad i, j \geqslant 1, \tag{3.1}$$

for some positive constant A. Under Assumption (3.1) it is proved in ref. 2, Theorems 2.4 and 2.5 that there is at least one density-conserving solution to (1.1) for every initial data in X^+ and the purpose of this section is to show that a similar result holds true for (1.2). The proof carried out in ref. 2 involves rather delicate estimates to control the tail of the series in (1.1) and an alternative proof based on estimates on generalised moments has been proposed in ref. 16. We shall here develop further this method and show that it applies to the study of (1.2).

3.1. Existence of Density-Conserving Solutions

The main result of this section is:

Theorem 3.1. Assume that $c^0 \in X^+$. Under the assumptions (1.4)–(1.5) and (3.1) there is at least one solution c to (1.2)–(1.3) on $[0,+\infty)$ satisfying

$$\|c(t)\|_X = \|c^0\|_X, \qquad t \in [0,+\infty). \tag{3.2}$$

In other words the density of the solution c is conserved throughout time evolution.

Before proceeding with the proof of Theorem 3.1 we need some preliminary results. We denote by $\mathscr{K}_1$ the set of non-negative and convex functions $U \in \mathscr{C}^1([0,+\infty)) \cap W^{2,\infty}_{loc}(0,+\infty)$ such that $U(0)=0$, $U'(0) \geqslant 0$ and U' is a concave function. We next denote by $\mathscr{K}_{1,\infty}$ the set of functions $U \in \mathscr{K}_1$ satisfying in addition

$$\lim_{r \to +\infty} U'(r) = \lim_{r \to +\infty} \frac{U(r)}{r} = +\infty. \tag{3.3}$$

Observe that $r \mapsto r^m$ belongs to $\mathscr{K}_1$ for $m \in [1,2]$ and to $\mathscr{K}_{1,\infty}$ for $m \in (1,2]$. We first recall the following lemma.

Lemma 3.2 (ref. 16, Lemma 3.2). For $U \in \mathscr{K}_1$ and $i, j \geqslant 1$ there holds

$$(i+j)(U(i+j)-U(i)-U(j)) \leqslant 2(iU(j)+jU(i)). \tag{3.4}$$

The next lemma is the main estimate needed to prove Theorem 3.1.

Lemma 3.3. Consider $T \in (0,+\infty)$ and $U \in \mathscr{K}_1$. There is a constant γ_T depending only on A, U, $\|c^0\|_X$ and T such that, for each $N \geqslant 3$, the solution c^N to (2.2)–(2.3) given by Lemma 2.2 satisfies

$$\sum_{i=1}^{N} U(i)\, c_i^N(t) \leqslant \gamma_T \sum_{i=1}^{N} U(i)\, c_i^0, \qquad t \in [0,T], \tag{3.5}$$

$$0 \leqslant \int_0^T \sum_{i=1}^{N-1} \sum_{j=1}^{N-i} \sum_{s=1}^{i+j-1} s\left(\frac{U(i+j)}{i+j} - \frac{U(s)}{s}\right) D_{i,j}^s(c^N(\tau))\, d\tau$$
$$\leqslant \gamma_T \sum_{i=1}^{N} U(i)\, c_i^0, \tag{3.6}$$

where $D_{i,j}^s(.)$ is defined by (2.1).

Proof. For $N \geqslant 3$ and $t \in [0,T]$ we put

$$M_U^N(t) = \sum_{i=1}^{N} U(i)\, c_i^N(t).$$

We infer from (2.4) and (3.1) that

$$\frac{dM_U^N}{dt} \leqslant \frac{A}{2} \sum_{i=1}^{N-1} \sum_{j=1}^{N-i} (i+j)(U(i+j) - U(i) - U(j))\, c_i^N c_j^N$$
$$- \frac{1}{2} \sum_{i=1}^{N-1} \sum_{j=1}^{N-i} (1-w_{i,j})\Big(U(i+j) - \sum_{s=1}^{i+j-1} N_{i,j}^s U(s)\Big)\, a_{i,j} c_i^N c_j^N.$$

By (1.5) we have

$$U(i+j) - \sum_{s=1}^{i+j-1} N_{i,j}^s U(s) = \sum_{s=1}^{i+j-1} s N_{i,j}^s \left(\frac{U(i+j)}{i+j} - \frac{U(s)}{s}\right).$$

The above two formulae, (2.5) and (3.4) now yield

$$\frac{dM_U^N}{dt} \leqslant A \sum_{i=1}^{N-1} \sum_{j=1}^{N-i} (iU(j) + jU(i))\, c_i^N c_j^N$$
$$- \frac{1}{2} \sum_{i=1}^{N-1} \sum_{j=1}^{N-i} \sum_{s=1}^{i+j-1} s\left(\frac{U(i+j)}{i+j} - \frac{U(s)}{s}\right) D_{i,j}^s(c^N),$$
$$\frac{dM_U^N}{dt} \leqslant 2A\, \|c^0\|_X\, M_U^N$$
$$- \frac{1}{2} \sum_{i=1}^{N-1} \sum_{j=1}^{N-i} \sum_{s=1}^{i+j-1} s\left(\frac{U(i+j)}{i+j} - \frac{U(s)}{s}\right) D_{i,j}^s(c^N). \tag{3.7}$$

Now, since $U(0)=0$ and U is a convex function, the function $r \mapsto U(r)/r$ is a non-decreasing function and the second term of the right-hand side of (3.7) is non-negative. Therefore

$$\frac{dM_U^N}{dt} \leqslant 2A \, \|c^0\|_X \, M_U^N,$$

which yields (3.5) by the Gronwall lemma. We next integrate (3.7) over $(0,T)$ and use (3.5) to obtain (3.6). ∎

We next derive a further estimate which entails the componentwise relative compactness of (c^N).

Lemma 3.4. Let $T \in (0,+\infty)$ and $i \geqslant 1$. There is a constant $\gamma_i(T)$ depending only on A, $\|c^0\|_X$, i and T such that, for each $N \geqslant \max(i,3)$,

$$\left\| \frac{dc_i^N}{dt} \right\|_{L^1(0,T)} \leqslant \gamma_i(T). \tag{3.8}$$

Proof. By (2.2) we have

$$\begin{aligned} 0 &\leqslant \frac{1}{2} \int_0^T \sum_{j=i+1}^{N} \sum_{k=1}^{j-1} D_{j-k,k}^i(c^N(\tau)) \, d\tau \\ &\leqslant \int_0^T \sum_{j=1}^{N-i} a_{i,j} c_i^N(\tau) \, c_j^N(\tau) \, d\tau + c_i^N(T). \end{aligned}$$

Owing to (3.1) and (2.5) we may estimate the right-hand side of the above inequality and obtain

$$\left\| \sum_{j=i+1}^{N} \sum_{k=1}^{j-1} D_{j-k,k}^i(c^N) \right\|_{L^1(0,T)} \leqslant 2AiT \, \|c^0\|_X^2 + 2 \, \|c^0\|_X.$$

The estimate (3.8) then follows from (2.2), (1.4), (3.1), (2.5) and the above estimate. ∎

We are now in a position to prove Theorem 3.1. For that purpose we first recall a refined version of the de la Vallée-Poussin theorem for integrable functions [ref. 19, Proposition I.1.1].

Theorem 3.5. Let $(\Omega, \mathscr{B}, \mu)$ be a measured space and consider $w \in L^1(\Omega, \mathscr{B}, \mu)$. Then there exists a function $V \in \mathscr{K}_{1,\infty}$ such that

$$V(|w|) \in L^1(\Omega, \mathscr{B}, \mu).$$

Remark 3.6. Theorem 3.5 is a classical result when $\mu(\Omega) < \infty$ (see, e.g., ref. 10, p. 38), except for the possibility of choosing V' concave. This last fact has been noticed in ref. 19.

Proof of Theorem 3.1. We apply Theorem 3.5, Ω being the set $\mathbb{N}\backslash\{0\}$ and $\mathscr{B}$ the set of all subsets of $\mathbb{N}\backslash\{0\}$. Defining the measure μ by

$$\mu(I) = \sum_{i \in I} c_i^0, \qquad I \subset \mathbb{N}\backslash\{0\},$$

the condition $c^0 \in X^+$ ensures that $x \mapsto x$ belongs to $L^1(\Omega, \mathscr{B}, \mu)$. By Theorem 3.5 there is thus a function $U_0 \in \mathscr{K}_{1,\infty}$ such that $x \mapsto U_0(x)$ belongs to $L^1(\Omega, \mathscr{B}, \mu)$, that is,

$$\mathscr{U}_0 := \sum_{i=1}^{\infty} U_0(i)\, c_i^0 < \infty. \tag{3.9}$$

In the following we denote by C any positive constant depending only on A, $\|c^0\|_X$, U_0 and $\mathscr{U}_0$. The dependence of C upon additional parameters will be indicated explicitly.

By (2.5) and (3.8) the sequence $(c_i^N)_{N \geqslant i}$ is bounded in $W^{1,1}(0,T)$ for each $i \geqslant 1$ and $T \in (0, +\infty)$. We then infer from the Helly theorem [ref. 13, pp. 372–374] that there are a subsequence of $(c_i^N)_{N \geqslant i}$, still denoted by $(c_i^N)_{N \geqslant i}$, and a sequence $c = (c_i)_{i \geqslant 1}$ of functions of locally bounded variation such that

$$\lim_{N \to +\infty} c_i^N(t) = c_i(t) \tag{3.10}$$

for each $i \geqslant 1$ and $t \geqslant 0$. Clearly $c_i(t) \geqslant 0$ for $i \geqslant 1$ and $t \geqslant 0$ and it follows from (3.10) and (2.5) that $c(t) \in X^+$ with

$$\|c(t)\|_X \leqslant \|c^0\|_X, \qquad t \geqslant 0. \tag{3.11}$$

Furthermore, as $U_0 \in \mathscr{K}_{1,\infty}$ we infer from (3.9) and Lemma 3.3 that, for each $T \geqslant 0$ and $N \geqslant 3$, there holds

$$\sum_{i=1}^{N} U_0(i)\, c_i^N(t) \leqslant C(T), \qquad t \in [0,T], \tag{3.12}$$

$$0 \leqslant \int_0^T \sum_{i=1}^{N-1} \sum_{j=1}^{N-i} \sum_{s=1}^{i+j-1} s \left(\frac{U_0(i+j)}{i+j} - \frac{U_0(s)}{s} \right) D_{i,j}^s(c^N(\tau))\, d\tau \leqslant C(T). \tag{3.13}$$

Owing to (3.10) the lower semicontinuity of the left-hand sides of (3.12)–(3.13) and the Fatou lemma allow to conclude that, for each $T \geqslant 0$,

$$\sum_{i=1}^{\infty} U_0(i)\, c_i(t) \leqslant C(T), \qquad t \in [0, T], \tag{3.14}$$

$$0 \leqslant \int_0^T \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \sum_{s=1}^{i+j-1} s \left(\frac{U_0(i+j)}{i+j} - \frac{U_0(s)}{s} \right) D_{i,j}^s(c(\tau))\, d\tau \leqslant C(T). \tag{3.15}$$

Let $i \geqslant 1$. On the one hand, since $U_0 \in \mathscr{K}_{1,\infty}$ it follows from (3.1) and (3.14) that

$$\sum_{j=1}^{\infty} a_{i,j} c_j \in L^1(0, T). \tag{3.16}$$

On the other hand we have

$$\sum_{j=i+1}^{\infty} \sum_{k=1}^{j-1} D_{j-k,k}^i(c) = \sum_{j+k \geqslant i+1} D_{j,k}^i(c), \tag{3.17}$$

and we infer from (3.15) and the properties of U_0 that

$$\sum_{j=i+1}^{\infty} \sum_{k=1}^{j-1} D_{j-k,k}^i(c) \in L^1(0, T). \tag{3.18}$$

Consider now $s \geqslant 1$ and $M \geqslant 2$. By (2.5), (3.10), (3.11) and the Lebesgue dominated convergence theorem we have

$$\lim_{N \to +\infty} \left| \sum_{j=1}^{M} a_{s,j} (c_s^N c_j^N - c_s c_j) \right|_{L^1(0,T)} = 0.$$

We next infer from (3.5), (2.5) and (3.1) that, for $N \geqslant s + M + 1$,

$$\begin{aligned} \left| \sum_{j=M+1}^{N-s} a_{s,j} c_s^N c_j^N \right|_{L^1(0,T)} &\leqslant 2As\, \|c^0\|_X \left| \sum_{j=M+1}^{N-s} j c_j^N \right|_{L^1(0,T)} \\ &\leqslant C(s,T) \sup_{j \geqslant M} \frac{j}{U_0(j)} \left| \sum_{j=M+1}^{N-s} U_0(j)\, c_j^N \right|_{L^1(0,T)} \end{aligned}$$

$$\left| \sum_{j=M+1}^{N-s} a_{s,j} c_s^N c_j^N \right|_{L^1(0,T)} \leqslant C(s,T) \sup_{j \geqslant M} \frac{j}{U_0(j)}.$$

Similarly (3.1), (3.11) and (3.14) yield

$$\left\| \sum_{j=M+1}^{\infty} a_{s,j} c_s c_j \right\|_{L^1(0,T)} \leqslant C(s,T) \sup_{j \geqslant M} \frac{j}{U_0(j)}.$$

Combining the above three estimates we obtain

$$\limsup_{N \to +\infty} \left\| \sum_{j=1}^{N-s} a_{s,j} c_s^N c_j^N - \sum_{j=1}^{\infty} a_{s,j} c_s c_j \right\|_{L^1(0,T)} \leqslant C(s,T) \sup_{j \geqslant M} \frac{j}{U_0(j)}.$$

The above inequality being valid for each $M \geqslant 2$ we use again the fact that $U_0 \in \mathscr{K}_{1,\infty}$ to conclude that

$$\lim_{N \to +\infty} \left\| \sum_{j=1}^{N-s} a_{s,j} c_s^N c_j^N - \sum_{j=1}^{\infty} a_{s,j} c_s c_j \right\|_{L^1(0,T)} = 0. \tag{3.19}$$

Finally consider $s \geqslant 1$ and $\varepsilon \in (0, 1)$. Since $U_0 \in \mathscr{K}_{1,\infty}$ there is $M \geqslant s+1$ such that

$$j \geqslant M \;\Rightarrow s \left(\frac{U_0(j)}{j} - \frac{U_0(s)}{s} \right) \geqslant \frac{1}{\varepsilon}. \tag{3.20}$$

For $N \geqslant s+1$ we have

$$\sum_{j=s+1}^{N} \sum_{k=1}^{j-1} D_{j-k,k}^s(c^N) = \sum_{s+1 \leqslant j+k \leqslant N} D_{j,k}^s(c^N).$$

Recalling (3.17) we see that

$$\sum_{j=s+1}^{N} \sum_{k=1}^{j-1} D_{j-k,k}^s(c^N) - \sum_{j=s+1}^{\infty} \sum_{k=1}^{j-1} D_{j-k,k}^s(c)$$

$$= \sum_{s+1 \leqslant j+k \leqslant N} D_{j,k}^s(c^N) - \sum_{j+k \geqslant s+1} D_{j,k}^s(c). \tag{3.21}$$

On the one hand it follows from (3.10), (2.5), (3.11) and the Lebesgue dominated convergence theorem that

$$\lim_{N \to +\infty} \left\| \sum_{s+1 \leqslant j+k \leqslant M} (D_{j,k}^s(c^N) - D_{j,k}^s(c)) \right\|_{L^1(0,T)} = 0. \tag{3.22}$$

On the other hand we have by (3.20)

$$D^s_{j,k}(c^N) \leqslant \varepsilon s \left(\frac{U_0(j+k)}{j+k} - \frac{U_0(s)}{s} \right) D^s_{j,k}(c^N)$$

for (j, k) such that $j+k \geqslant M+1$, hence by (3.6),

$$\left| \sum_{M+1 \leqslant j+k \leqslant N} D^s_{j,k}(c^N) \right|_{L^1(0,T)} \leqslant C\varepsilon. \tag{3.23}$$

Similarly it follows from (3.20) and (3.15) that

$$\left| \sum_{M+1 \leqslant j+k} D^s_{j,k}(c) \right|_{L^1(0,T)} \leqslant C\varepsilon. \tag{3.24}$$

Combining (3.21)–(3.24) yields

$$\limsup_{N \to +\infty} \left| \sum_{j=s+1}^{N} \sum_{k=1}^{j-1} D^s_{j-k,k}(c^N) - \sum_{j=s+1}^{\infty} \sum_{k=1}^{j-1} D^s_{j-k,k}(c) \right|_{L^1(0,T)} \leqslant C\varepsilon$$

for each $\varepsilon \in (0, 1)$. Consequently

$$\lim_{N \to +\infty} \left| \sum_{j=s+1}^{N} \sum_{k=1}^{j-1} D^s_{j-k,k}(c^N) - \sum_{j=s+1}^{\infty} \sum_{k=1}^{j-1} D^s_{j-k,k}(c) \right|_{L^1(0,T)} = 0. \tag{3.25}$$

Owing to (3.10), (2.5), (3.11), (3.19) and (3.25) it is now straightforward to check that c_i satisfies Definition 2.1(ii) for each $i \geqslant 1$. By (3.16), (3.18) and (3.11) the right-hand side of the identity in Definition 2.1(ii) is integrable over $(0, t)$ for each $t > 0$, and the continuity of c_i follows. We have thus shown that $c = (c_i)$ is a solution to (1.2)–(1.3) on $[0, +\infty)$. In order to complete the proof of Theorem 3.1 it remains to check (3.2). Let $t \in (0, +\infty)$. For $N \geqslant M \geqslant 3$ we have by (2.5)

$$\begin{aligned} \big| \|c(t)\|_X - \|c^0\|_X \big| \leqslant & \sum_{i=1}^{M} i \, |c_i^N(t) - c_i(t)| + \sum_{i=N+1}^{\infty} i c_i^0 \\ & + \sum_{i=M+1}^{N} i c_i^N(t) + \sum_{i=M+1}^{\infty} i c_i(t). \end{aligned}$$

It then follows from (3.12) and (3.14) that

$$\left| \|c(t)\|_X - \|c^0\|_X \right| \leqslant \sum_{i=1}^{M} i\,|c_i^N(t) - c_i(t)| + \sum_{i=N+1}^{\infty} i c_i^0 + C(T) \sup_{i \geqslant M} \frac{i}{U_0(i)}.$$

Since $c^0 \in X^+$ we first deduce from (3.10) that

$$\left| \|c(t)\|_X - \|c^0\|_X \right| \leqslant C(T) \sup_{i \geqslant M} \frac{i}{U_0(i)}.$$

Recalling that $U_0 \in \mathscr{K}_{1,\infty}$ we conclude that $\|c(t)\|_X = \|c^0\|_X$ and the proof of Theorem 3.1 is complete. ∎

3.2. Propagation of Moments and Uniqueness

The question we consider here is whether, given $c^0 \in X^+$ such that $\sum i^m c_i^0 < \infty$ for some $m > 1$, the solution c to (1.2)–(1.3) constructed in Theorem 3.1 enjoys the same properties throughout time evolution, that is, $\sum i^m c_i(t) < \infty$ for $t \in (0, +\infty)$. This question has a positive answer for the discrete coagulation equations with spontaneous breakage (1.1)[4, 5] and our next result states that the answer is also positive for (1.2).

Proposition 3.7. Assume that the assumptions (1.4)–(1.5) and (3.1) are fulfilled, and consider $c^0 \in X^+$ such that

$$\sum_{i=1}^{\infty} i^m c_i^0 < \infty \tag{3.26}$$

for some $m > 1$. Then the solution c to (1.2)–(1.3) on $[0, +\infty)$ constructed in Theorem 3.1 satisfies

$$\sup_{t \in [0,T]} \sum_{i=1}^{\infty} i^m c_i(t) < \infty$$

for each $T > 0$.

Proof. By (3.10) we know that

$$\lim_{N \to +\infty} c_i^N(t) = c_i(t)$$

for each $t \in [0,+\infty)$ and $i \geqslant 1$, where c^N still denotes the solution to (2.2)–(2.3) given by Lemma 2.2. Taking $g_i = i^m$ in (2.4) we obtain

$$\frac{d}{dt} \sum_{i=1}^{N} i^m c_i^N \leqslant \frac{1}{2} \sum_{i=1}^{N-1} \sum_{j=1}^{N-i} ((i+j)^m - i^m - j^m)\, a_{i,j} c_i^N c_j^N,$$

since the convexity of $r \mapsto r^m$ entails that

$$(i+j)^m \geqslant \sum_{s=1}^{i+j-1} N_{i,j}^s s^m, \qquad i, j \geqslant 1.$$

Now, by ref. 4, Lemma 2.3 there is a constant κ_m depending only on m such that

$$(i+j)((i+j)^m - i^m - j^m) \leqslant \kappa_m (i j^m + j i^m), \qquad i, j \geqslant 1.$$

It follows from (3.1), (2.5) and the above inequality that

$$\begin{aligned} \frac{d}{dt} \sum_{i=1}^{N} i^m c_i^N &\leqslant \frac{A\kappa_m}{2} \sum_{i=1}^{N-1} \sum_{j=1}^{N-i} (i j^m + j i^m)\, c_i^N c_j^N \\ &\leqslant A\kappa_m \, \|c^0\|_X \sum_{i=1}^{N} i^m c_i^N, \end{aligned}$$

and the Gronwall lemma yields

$$\sum_{i=1}^{N} i^m c_i^N(t) \leqslant \exp(A\kappa_m \, \|c^0\|_X \, t) \sum_{i=1}^{N} i^m c_i^0, \qquad t \geqslant 0.$$

Owing to (3.26) and (3.10) we may pass to the limit as $N \to +\infty$ in the above inequality and obtain

$$\sum_{i=1}^{\infty} i^m c_i(t) \leqslant \exp(A\kappa_m \, \|c^0\|_X \, t) \sum_{i=1}^{\infty} i^m c_i^0, \qquad t \geqslant 0.$$

The proof of Proposition 3.7 is thus complete. ∎

As a consequence of Proposition 3.7 we may prove the following uniqueness result.

Proposition 3.8. Assume that the assumptions (1.4)–(1.5) are fulfilled and there are $\alpha \in [0, 1]$ and $K_\alpha > 0$ such that

$$a_{i,j} \leqslant K_\alpha (i^\alpha + j^\alpha), \qquad i, j \geqslant 1. \tag{3.27}$$

Consider next $c^0 \in X^+$ such that

$$\sum_{i=1}^{\infty} i^{1+\alpha} c_i^0 < \infty. \tag{3.28}$$

Then there is a unique solution c to (1.2)–(1.3) on $[0, +\infty)$ satisfying both (3.2) and

$$\sup_{t \in [0,T]} \sum_{i=1}^{\infty} i^{1+\alpha} c_i(t) < \infty \tag{3.29}$$

for each $T \in (0, +\infty)$.

Proof. As $\alpha \in [0, 1]$ it follows from (3.27) that $(a_{i,j})$ satisfy (3.1) and the existence of a solution to (1.2)–(1.3) on $[0, +\infty)$ with the properties stated in Proposition 3.8 is a consequence of Theorem 3.1 and Proposition 3.7.

As for uniqueness we follow closely the approach developed in ref. 2, Theorem 4.2. Given $c^0 \in X^+$ satisfying (3.28) we consider two solutions $c = (c_i)$ and $\hat{c} = (\hat{c}_i)$ to (1.2)–(1.3) on $[0, +\infty)$ enjoying the property (3.29). For $i \geqslant 1$ we put

$$z_i = c_i - \hat{c}_i \qquad \text{and} \qquad \sigma_i = sign(z_i),$$

where $sign(r) = r/|r|$ if $r \in \mathbb{R} \backslash \{0\}$ and $sign(0) = 0$. Fix $n \geqslant 2$ and $t \in (0, T)$. We infer from (1.2) that

$$\sum_{i=1}^{n} i\, |z_i(t)| = \int_0^t \sum_{i=1}^{4} \mathcal{S}_i^n(\tau)\, d\tau, \tag{3.30}$$

where

$$\mathcal{S}_1^n = \frac{1}{2} \sum_{i=1}^{n-1} \sum_{j=1}^{n-i} w_{i,j}((i+j)\, \sigma_{i+j} - i\sigma_i - j\sigma_j)\, a_{i,j}(c_i c_j - \hat{c}_i \hat{c}_j),$$

$$\mathcal{S}_2^n = \frac{1}{2} \sum_{i=1}^{n-1} \sum_{j=1}^{n-i} (1 - w_{i,j}) \left(\sum_{s=1}^{i+j-1} s\sigma_s N_{i,j}^s - i\sigma_i - j\sigma_j \right) a_{i,j}(c_i c_j - \hat{c}_i \hat{c}_j),$$

$$\mathcal{S}_3^n = -\sum_{i=1}^{n} \sum_{j=n+1-i}^{\infty} i\sigma_i a_{i,j}(c_i c_j - \hat{c}_i \hat{c}_j),$$

$$\mathcal{S}_4^n = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=n+1}^{\infty} \sum_{k=1}^{j-1} i\sigma_i (D_{j-k,k}^i(c) - D_{j-k,k}^i(\hat{c})).$$

Noticing that

$$((i+j)\,\sigma_{i+j}-i\sigma_i-j\sigma_j)\,z_i=((i+j)\,\sigma_{i+j}\sigma_i-i-j\sigma_j\sigma_i)\,|z_i|$$
$$\leqslant 2j\,|z_i|,$$

the first term $\mathscr{S}_1^n$ can be estimated as follows:

$$\mathscr{S}_1^n\leqslant \sum_{i=1}^{n-1}\sum_{j=1}^{n-i} w_{i,j}a_{i,j}(\,jc_j\,|z_i|+i\hat{c}_i\,|z_j|),$$

hence by (3.27) and (1.4)

$$\mathscr{S}_1^n\leqslant 2K_\alpha\left(\sum_{i=1}^{n} i^{1+\alpha}(c_i+\hat{c}_i)\right)\sum_{i=1}^{n} i\,|z_i|. \tag{3.31}$$

Similarly we have by (1.5)

$$\left(\sum_{s=1}^{i+j-1} s\sigma_s N_{i,j}^s-i\sigma_i-j\sigma_j\right)z_i=\left(\sum_{s=1}^{i+j-1} s\sigma_s\sigma_i N_{i,j}^s-i-j\sigma_j\sigma_i\right)|z_i|$$
$$\leqslant 2j\,|z_i|.$$

Consequently, using once more (3.27) and (1.4) we obtain

$$\mathscr{S}_2^n\leqslant 2K_\alpha\left(\sum_{i=1}^{n} i^{1+\alpha}(c_i+\hat{c}_i)\right)\sum_{i=1}^{n} i\,|z_i|. \tag{3.32}$$

We next infer from (3.27) that

$$\int_0^t\left|\sum_{i=1}^{n}\sum_{j=n+1-i}^{\infty} i\sigma_i a_{i,j}c_ic_j\right|d\tau\leqslant K_\alpha\int_0^t\sum_{i=1}^{n}\sum_{j=n+1-i}^{\infty}(i^{1+\alpha}+ij^\alpha)\,c_ic_j\,d\tau,$$

and it follows at once from (3.29) that

$$\lim_{n\to+\infty}\int_0^t\left|\sum_{i=1}^{n}\sum_{j=n+1-i}^{\infty} i\sigma_i a_{i,j}c_ic_j\right|d\tau=0.$$

As a similar result is available for $\hat{c}$ we conclude that

$$\lim_{n\to+\infty}\mathscr{S}_3^n=0. \tag{3.33}$$

Finally it follows from (1.2) that

$$\sum_{i=1}^{n} ic_i(t) = \sum_{i=1}^{n} ic_i^0 - \int_0^t \sum_{i=1}^{n} \sum_{j=n+1-i}^{\infty} ia_{i,j} c_i c_j \, d\tau + \frac{1}{2} \int_0^t \sum_{i=1}^{n} \sum_{j=n+1}^{\infty} \sum_{k=1}^{j-1} iD_{j-k,k}^i(c) \, d\tau.$$

Now c is a density-conserving solution to (1.2) and an argument similar to the proof of (3.33) ensures that the second term of the right-hand side of the above identity converges to zero as $n \to +\infty$. Letting $n \to +\infty$ then yields

$$\lim_{n \to +\infty} \int_0^t \sum_{i=1}^{n} \sum_{j=n+1}^{\infty} \sum_{k=1}^{j-1} iD_{j-k,k}^i(c) \, d\tau = 0,$$

from which we easily deduce that

$$\lim_{n \to +\infty} \mathcal{S}_4^n = 0. \tag{3.34}$$

Owing to (3.29) and (3.31)–(3.34) we may pass to the limit as $n \to +\infty$ in (3.30) and obtain

$$\sum_{i=1}^{n} i \, |z_i(t)| \leqslant 4K_\alpha \int_0^t \left(\sum_{i=1}^{n} i^{1+\alpha}(c_i + \hat{c}_i) \right) \sum_{i=1}^{n} i \, |z_i| \, d\tau.$$

Applying the Gronwall lemma then completes the proof of Proposition 3.8. ∎

4. REMARKS ON LARGE TIME BEHAVIOUR AND GELATION

We end up this paper with the study of the large time behaviour of solutions to some particular cases of (1.2)–(1.3). We begin with the nonlinear breakage model (1.7).[8] As already mentioned, in this model, a cluster only produces fragments of smaller sizes after collision. We thus expect that only 1-clusters remain in the long time. More precisely we have the following result:

Proposition 4.1. Assume that $(a_{i,j})$ satisfy (1.4) and (3.1), $w_{i,j} \equiv 0$ and

$$N_{i,j}^s = \mathbf{1}_{[s,+\infty)}(i) \, b_{s,i;j} + \mathbf{1}_{[s,+\infty)}(j) \, b_{s,j;i}, \tag{4.1}$$

where $\{b_{i,j;k},\ 1 \leqslant i \leqslant j-1\}$ denotes the distribution function of the fragments of a j-cluster after a collision with a k-cluster, and satisfies

$$\sum_{i=1}^{j-1} i b_{i,j;k} = j, \qquad j \geqslant 2,\ k \geqslant 1. \tag{4.2}$$

For $c^0 \in X^+$ there is a density-conserving solution c to (1.2)–(1.3) on $[0, +\infty)$ and there is $c^\infty = (c_i^\infty) \in X^+$ such that

$$\lim_{t \to +\infty} \|c(t) - c^\infty\|_X = 0. \tag{4.3}$$

Moreover, if $i \geqslant 2$ is such that $a_{i,i} \neq 0$ we have

$$c_i^\infty = 0. \tag{4.4}$$

Remark 4.2. In particular, if $a_{i,i} > 0$ for each $i \geqslant 2$ then $c_i^\infty = 0$ for every $i \geqslant 2$ and (3.2) and (4.3) entail that $c_1^\infty = \|c^0\|_X$.

Proof. First, as $(a_{i,j})$ satisfy (3.1) the existence of a density-conserving solution follows from Theorem 3.1. Consider next $m \geqslant 1$, $t_1 \geqslant 0$ and $t_2 \geqslant t_1$. We multiply the ith equation of (1.2) by i and sum the resulting identities from $i = 1$ to $i = m$. After integrating over (t_1, t_2) and some calculations we obtain

$$\begin{aligned}
\sum_{i=1}^{m} i(c_i(t_2) - c_i(t_1)) &= \frac{1}{2} \int_{t_1}^{t_2} \sum_{j=1}^{m} \sum_{k=m+1-j}^{m} \left(\sum_{i=1}^{m} i N_{j,k}^i - j - k \right) a_{j,k} c_j c_k \, d\tau \\
&\quad + \int_{t_1}^{t_2} \sum_{j=1}^{m} \sum_{k=m+1}^{\infty} \left(\sum_{i=1}^{m} i N_{j,k}^i - j \right) a_{j,k} c_j c_k \, d\tau \\
&\quad + \frac{1}{2} \int_{t_1}^{t_2} \sum_{j=m+1}^{\infty} \sum_{k=m+1}^{\infty} \sum_{i=1}^{m} i D_{j,k}^i(c) \, d\tau.
\end{aligned} \tag{4.5}$$

On the one hand (4.1)–(4.2) entail that

$$\sum_{i=1}^{m} i N_{j,k}^i = \sum_{i=1}^{j} i b_{i,j;k} + \sum_{i=1}^{k} i b_{i,k;j} = j + k$$

for $j \in \{1, \ldots, m\}$ and $k \in \{m+1-j, \ldots, m\}$, and the first term of the right-hand side of (4.5) is equal to zero. On the other hand, using again (4.1)–(4.2), we obtain

$$\sum_{i=1}^{m} i N_{j,k}^i = \sum_{i=1}^{j} i b_{i,j;k} + \sum_{i=1}^{m} i b_{i,k;j} = j + \sum_{i=1}^{m} i b_{i,k;j}$$

for $j \in \{1,\dots, m\}$ and $k \geqslant m+1$, hence a non-negative bound from below for the second term of the right-hand side of (4.5). Therefore (4.5) yields

$$\sum_{i=1}^{m} i(c_i(t_2)-c_i(t_1)) \geqslant \frac{1}{2}\int_{t_1}^{t_2} \sum_{j=1}^{m} \sum_{k=m+1}^{\infty} \sum_{i=1}^{m} ib_{i,k;j}a_{j,k}c_jc_k \, d\tau + \frac{1}{2}\int_{t_1}^{t_2} \sum_{j=m+1}^{\infty} \sum_{k=m+1}^{\infty} \sum_{i=1}^{m} ib_{i,k;j}a_{j,k}c_jc_k \, d\tau.$$

$$\sum_{i=1}^{m} i(c_i(t_2)-c_i(t_1)) \geqslant \frac{1}{2}\int_{t_1}^{t_2} \sum_{j=1}^{\infty} \sum_{k=m+1}^{\infty} \sum_{i=1}^{m} ib_{i,k;j}a_{j,k}c_jc_k \, d\tau. \tag{4.6}$$

The first consequence of (4.6) is that the function

$$S_m : t \mapsto \sum_{i=1}^{m} ic_i(t) \text{ is a non-decreasing function on } [0,+\infty). \tag{4.7}$$

Owing to the conservation of density the function S_m is also bounded from above and we conclude that $S_m(t)$ has a limit as $t \to +\infty$ for each $m \geqslant 1$. Recalling that $c_m(t) = (S_m(t) - S_{m-1}(t))/m$ we readily obtain that there is a non-negative real number c_m^∞ such that

$$\lim_{t \to +\infty} c_m(t) = c_m^\infty, \qquad m \geqslant 1. \tag{4.8}$$

Furthermore, as $c(t) \in X^+$ for each $t \geqslant 0$ the convergence (4.8) ensures that $c^\infty := (c_m^\infty)$ belongs to X^+. Also the density conservation and (4.7) entail that

$$\sum_{i=m}^{\infty} ic_i(t) \leqslant \sum_{i=m}^{\infty} ic_i^0, \qquad m \geqslant 1, \qquad t \geqslant 0.$$

This last fact and (4.8) yield (4.3).

Finally, another consequence of (4.6) and (4.3) is that

$$\int_0^\infty \sum_{j=1}^{\infty} \sum_{k=m+1}^{\infty} \sum_{i=1}^{m} ib_{i,k;j}a_{j,k}c_jc_k \, d\tau < \infty.$$

Let $i \geqslant 2$ such that $a_{i,i} > 0$. Then the above estimate with $m = i-1$ and $j = k = i$ asserts that

$$\sum_{s=1}^{m} sb_{s,i;i}a_{i,i}c_i^2 = ia_{i,i}c_i^2 \in L^1(0,+\infty).$$

Recalling (4.8) we obtain that $a_{i,i}(c_i^\infty)^2 = 0$, hence (4.4). ∎

Another example of a model involving only collisional breakage (without coagulation) is the case of coefficients satisfying a *detailed balance* condition of the form

$$N^{i}_{j-k,k} a_{j-k,k} Q_{j-k} Q_k = N^{k}_{j-i,i} a_{j-i,i} Q_{j-i} Q_i \tag{4.9}$$

for $j \geqslant 1$ and $1 \leqslant i, k \leqslant j-1$, where (Q_i) is a sequence of non-negative real numbers. The condition (4.9) amounts to assume a kind of reversibility in the collision interactions which could happen here thanks to the possible transfer of matter during collisions. More precisely, the number of k-clusters produced by the collision of an i-cluster with a $(j-i)$-cluster has to be balanced by the number of i-clusters resulting from the collision of a k-cluster with a $(j-k)$-cluster. For instance such a condition is fulfilled (with $Q_i \equiv 1$) by

$$N^{s}_{i,j} = \frac{2}{i+j-1}, \qquad a_{i,j} = (i+j)^{\omega}, \qquad \omega \in [0, 1].$$

Under the assumption (4.9), and suitable assumptions on (Q_i) as well, non-trivial steady states exist and it is expected that the solutions to (1.2)–(1.3) converge to a steady state. We refer to the forthcoming paper[18] for some results in that direction. It is worth mentioning here that the above mentioned results do not cover all the possible models (1.2) involving only collisional breakage (without coagulation).

For the general model (1.2)–(1.3) with both coagulation and collisional fragmentation the analysis of the large time behaviour of the solutions seems to be harder. Besides the case considered by Srivastava[24] we are able to show the convergence to steady states of solutions to (1.2)–(1.3) under the strong assumption that each collision has to involve a 1-cluster.[17] Such an assumption is reminiscent of the Becker–Döring model.[3] More precisely, we assume that the collision of an i-cluster with a 1-cluster results in either an $(i+1)$-cluster (coagulation) or an $(i-1)$-cluster and two 1-clusters (fragmentation). In the coagulation-dominating case ($w_{i,1} \geqslant 1/2$) and in the fragmentation-dominating case ($w_{i,1} \leqslant 1/2$) stabilization to a steady state is proved in ref. 17.

Let us finally point out that gelation can occur in the discrete coagulation equations with collisional breakage. More precisely we have the following result.

Proposition 4.3. Assume that $(a_{i,j})$, $(w_{i,j})$ and $(N^{s}_{i,j})$ satisfy (1.4)–(1.5) and

$$\lambda ij \leqslant w_{i,j} a_{i,j} \qquad \text{and} \qquad a_{i,j} \leqslant \Lambda ij, \tag{4.10}$$

$$\sum_{s=1}^{i+j-1} N_{i,j}^s = 2 \tag{4.11}$$

for $i, j \geqslant 1$, the constants λ and Λ being two positive real numbers.

Consider $c^0 \in X^+$, $c^0 \not\equiv 0$ and assume that (1.2)–(1.3) has a solution c on $[0, +\infty)$ such that $t \mapsto \|c(t)\|_X$ is a non-increasing function on $[0, +\infty)$. Then

$$\lim_{t \to +\infty} \|c(t)\|_X = 0.$$

In particular, $\|c(t)\|_X < \|c^0\|_X$ for t large enough, hence the occurrence of gelation.

Proof. We follow the lines of the proof of ref. 15, Proposition 5.1. We infer from (1.2) and (4.11) that, for $m \geqslant 1$, $t_1 \geqslant 0$ and $t_2 > t_1$ we have

$$\begin{aligned}\sum_{i=1}^{m} (c_i(t_2) - c_i(t_1)) = & -\frac{1}{2} \int_{t_1}^{t_2} \sum_{i=1}^{m-1} \sum_{j=1}^{m-i} a_{i,j} c_i c_j \, d\tau \\ & - \int_{t_1}^{t_2} \sum_{i=1}^{m} \sum_{j=m+1-i}^{\infty} a_{i,j} c_i c_j \, d\tau \\ & + \frac{1}{2} \int_{t_1}^{t_2} \sum_{i=1}^{m-1} \sum_{j=1}^{m-i} (1 - w_{i,j}) \, a_{i,j} c_i c_j \, d\tau \\ & + \int_{t_1}^{t_2} \sum_{i=1}^{m} \sum_{j=m+1}^{\infty} \sum_{k=1}^{j-1} D_{j-k,k}^i(c) \, d\tau.\end{aligned}$$

As $c(\tau)$ belongs to X^+ with $\|c(\tau)\|_X \leqslant \|c^0\|_X$ for every $\tau \in [t_1, t_2]$ the growth conditions (4.10)–(4.11) and (1.4) allow to pass to the limit as $m \to +\infty$ in the above equality ; we thus obtain:

$$\sum_{i=1}^{\infty} (c_i(t_2) - c_i(t_1)) \leqslant -\frac{1}{2} \int_{t_1}^{t_2} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} w_{i,j} a_{i,j} c_i c_j \, d\tau.$$

Using the lower bound in (4.10) we finally arrive at

$$\sum_{i=1}^{\infty} c_i(t_2) + \frac{\lambda}{2} \int_{t_1}^{t_2} \|c(\tau)\|_X^2 \, d\tau \leqslant \sum_{i=1}^{\infty} c_i(t_1).$$

Now consider $t \in (0, +\infty)$. As $t \mapsto \|c(t)\|_X$ is non-increasing we deduce from the previous estimate (with $t_1 = 0$ and $t_2 = t$) that

$$\frac{\lambda t}{2} \|c(t)\|_X^2 \leqslant \sum_{i=1}^{\infty} c_i^0 \leqslant \|c^0\|_X.$$

Thus

$$\|c(t)\|_X \leqslant \left(\frac{2\,\|c^0\|_X}{\lambda t} \right)^{-1/2}, \qquad t \in (0, +\infty),$$

and the proof of Proposition 4.3 is complete. ∎

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